Study of Aircraft Cruise

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The long-range aircraft cruise problem is analyzed using a model intermediate in complexity between the energy model and the point-mass model. It is shown that this formulation imbeds the classical steady-state cruise as the central member along with several other oscillatory extremals. Oscillatory cruise trajectories are shown to exist if the Hessian of a characteristic function is positive definite. An expression for predicting the frequency of oscillation in the neighborhood of the classical steady-state cruise point is developed. Qualitative effects of increasing the vehicle thrust and improving the lift-over-drag ratio are discussed. Numerical results for two fighter aircraft and a transport aircraft are given. While oscillatory cruise mode exists for the two fighter aircraft, steady-state cruise at full throttle is found to be optimal for the transport aircraft. A second variation analysis to bring out the reason for fuel savings along oscillatory trajectories is developed. It is shown that whenever the Hessian of the characteristic function is positive definite, the second variation will be zero along the classical steady-state cruise arc, indicating that a neighboring oscillatory extremal is competitive. Comparisons with the previous point-mass and energy model results are given.

Introduction

LONG-RANGE minimum fuel aircraft cruise remains as an interesting problem in the aircraft trajectory optimization area beginning with a series of papers by Zagalsky et al.,[1] Schultz and Zagalsky,[2] Speyer,[3,5] and Schultz.[4] In Ref. 1 the authors examined the long-range optimal aircraft cruise problem with the energy-range model and found that the extended velocity set for this problem is nonconvex. As a result, optimal controls may not exist in the class of piecewise continuous control functions.[6] However, relaxed controls in the class of measurable functions can exist. According to Speyer,[5] this was first noted by Edelbaum.[7]

Schultz and Zagalsky[2] next employed the intermediate vehicle model with throttle and flight-path angle as controls and showed that the steady-state cruise arc satisfies the Euler's necessary conditions. With the intermediate vehicle model, the control variables appear linearly in the Hamiltonian.[8] Since both control variables are interior to their bounds along the steady-state cruise trajectory, it is a doubly singular arc in the calculus of variations. It is known that along a minimizing singular arc, the generalized Legendre-Clebsch necessary condition, also known as the Kelley-Contensou test or the Goh-Robbins test, should be satisfied.[9,10] For the intermediate vehicle model considered in Ref. 2, Speyer[3] showed that the steady-state cruise arc does not satisfy the Goh-Robbins test. As a result, with this modeling, the steady-state cruise arc cannot be optimal.

Schultz,[4] using a point-mass model for aircraft flight, showed that the Kelley-Contensou test is satisfied along the steady-state cruise arc. Thus, at least over sufficiently short intervals, the steady-state cruise arc appeared to be optimal. The control variables in this model are the lift coefficient and throttle. In response to this work, Speyer,[5] using a point-mass model, Goh's transformation,[11] and a frequency domain version of the Jacobi test, showed that conjugate points occur along the steady-state cruise arc. As a result, the steady-state cruise arc is nonoptimal over long ranges. Moreover, for a class of aerodynamic models, including the one used by Schultz,[4] Speyer showed that the second variation can be made negative for a range of frequencies, the independent variable in his Jacobi test. This fact implies that an oscillatory trajectory about the steady-state cruise arc may provide a lower value of the performance index. In fact, Speyer computed the fuel savings for small amplitude oscillations about the steady-state cruise arc for a hypothetical aircraft. For this aircraft, the period of oscillation turns out to be approximately 2.8 min. This linearized analysis predicts an improvement in fuel consumption between 0.08 and 0.4%.

While the inclusion of complete vehicle dynamics is desirable in these studies, it is difficult to obtain analytical results because of model complexity. In order to gain a better understanding of the vehicle parameters responsible for the nonoptimality of steady-state cruise, Gilbert,[12] Gilbert and Parsons,[13] and Houlihan et al.[14] have studied this problem using reduced-order models. In particular, Refs. 13 and 14 approach this problem using the energy-range model, for which the extended velocity set or the hodograph is known to be nonconvex.[1,7] In the absence of hodograph convexity, an optimal control in the class of piecewise continuous functions may not exist. While there exists an infimum of the performance index, no piecewise continuous control function can produce that value. It then becomes necessary to extend the admissible control set to include functions that are only measurable,[6,15] in the sense of ordinary Lebesque measure.[8] The optimal control in such a relaxed problem is a chattering arc. For the aircraft cruise problem with the energy-range model, chattering controls can be constructed by examining the hodograph figure at several energy levels. Using this approach, Gilbert and Parsons[13] found that the chattering cruise produces about 1.7% fuel savings at the cruise energy for the F-4 aircraft. At lower energy levels, the difference between chattering cruise and the steady-state cruise is much higher, of the order of 7.3%. This led them to conclude that if a maximum altitude constraint was imposed such that the aircraft is forced to fly at lower energy levels, one would realize a more substantial improvement in fuel consumption. However, this does not imply that there would be any fuel savings with respect to the unconstrained classical steady-state cruise conditions.

Houlihan et al.[14] analyzed the hodograph of three aircraft configurations and concluded that, at the cruise energy, chattering cruise provides, at best, 5% fuel savings when com-

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pared with the steady-state cruise arc. Assuming constant air density, maximum thrust, and fuel flow rate, they showed that the ratio of maximum thrust to minimum drag controls the amount of fuel savings realized by employing chattering cruise. More interestingly, they obtained an expression for the local convexity of the hodograph at and near the power-off point, which turns out to be equivalent to the Goh-Robbins test obtained by Speyer.[3] Whereas the elegance of this approach is unquestionable, the chattering control functions generated are impossible to implement in any real aircraft, let alone evaluate in a simulation. This is because the analysis requires the aircraft to jump from one altitude-airspeed pair to another in close to zero time, at constant energy. It should be noted that the integrated effect of drag during this high-rate transition needs to be accounted for before predicting the expected fuel savings. Recently, Bilimoria et al.[16] approached the minimum time-fuel problem using a similar framework and noted the existence of chattering modes in certain altitude-airspeed regimes.

Another interesting approach to the aircraft cruise problem was advanced by Breakwell and Shoee.[17] Using a point-mass model for aircraft flight, they expanded the variational Hamiltonian to second order and identified a term that may be responsible for fuel savings along oscillatory cruise trajectories. By using sinusoidal altitude-airspeed trajectories, they estimate the fuel savings to be in the 0.5–3% range. This requires between 7,500 and 16,400 ft altitude oscillation for the particular drag models considered. It is important to stress that they assumed a constant thrust specific fuel consumption in their analysis.

Recent work on the long-range aircraft cruise problem has been oriented toward obtaining numerical solutions. For example, Gilbert and Lyons,[18] using a periodic spline parameterization of the state-control histories in conjunction with a nonlinear programming algorithm, computed the oscillatory cruise trajectories for a hypothetical aircraft. Speyer et al.[19] used a multiple-shooting method with periodicity assumptions to compute the oscillatory cruise trajectories for a hypersonic vehicle. Grimm et al.[20] employed a multiple-shooting algorithm with periodic boundary conditions to compute the oscillatory cruise trajectories for a version of the F-4 aircraft. Chuang and Speyer,[21] using the data for a hypersonic scramjet aircraft with such effects as engine-off drag penalty, thrust given as a function of Mach number, altitude, and angle of attack, computed oscillatory cruise trajectories. In all the numerical results given in Refs. 18–21, the throttle appears to be purely bang-bang, and all of them used the explicit assumption that cruise arc is periodic. Since the point-mass model permits singular arcs in the cruise problem because of the satisfaction of the Kelly-Contensou test for partial throttle,[4] there appears to be no reason why the throttle cannot be within its bounds at least during a part of the oscillatory cruise trajectory. This aspect of the cruise problem is yet to receive attention.

While the previously cited research was in progress, the investigators working in the transport aircraft area employed engineering assumptions to make the cruise problem more tractable. Notably, Barman and Erzberger[22] used the assumption that the optimal trajectory consists of a climb during which energy increases monotonically, a cruise leg during which the energy is constant, and a descent leg in which the energy decreases monotonically, synthesized an algorithm to compute minimum fuel, minimum time, and minimum direct operating cost trajectories for a short-haul aircraft. Using a similar set of assumptions, Erzberger and Lee[23] developed an algorithm suitable for use in airline flight planning and for onboard performance management. There is an important difference in vehicle modeling employed in these studies with those in Refs. 1–21; viz, the throttle control variable does not appear linearly in the vehicle model in the studies given in Refs. 22 and 23. Finally, it is worthwhile examining the research reported by Hargraves et al.[24] In that work, a Chebychev parameterized nonlinear programming algorithm was developed for solving optimal control problems. Specifically, Ref. 24 presents some results for long-range transport aircraft cruise problem assuming a thrust-throttle characteristic consisting of a linear and a quadratic term with an attached small parameter. The long-range trajectories for several values of the small parameter are given in that research. The lowest value of the parameter used was 0.01 and, for this value, long-range cruise turns out to be steady-state flight at full throttle. Interestingly, in the work reported by Barman and Erzberger,[22] the throttle approaches the maximum value whenever the component of cost associated with the flight time is made zero, viz, $\sigma = 1$, in their direct operating cost criterion. It appears that, if the maximum engine pressure ratio constraint was not included in the computations, the throttle may indeed have been at its maximum value.

With this background, in the present study, an intermediate vehicle model[2] will be used to study the aircraft cruise problem. Some new results and connections with earlier reported results using the point-mass model and the energy-range model will be given. Numerical results for three aircraft, two high-performance fighters, and a transport aircraft, will be presented. Conditions for the existence of oscillatory cruise extremals as well as the possible reasons for fuel savings along oscillatory cruise paths will be advanced. A second variation analysis along the classical steady-state cruise arc also will be given.

Vehicle Modeling

The intermediate vehicle model for aircraft symmetric flight can be obtained by assuming that the flight-path angle and its rate are small in the point-mass model.[25] This assumption permits the replacement of $\sin\gamma$, $\cos\gamma$ terms appearing in the point-mass model by their small-angle approximations. This is reasonable in the aircraft cruise analysis because one does not expect large path angles because of the fuel penalty involved. An alternate form of the intermediate vehicle model can be obtained by retaining the $\sin\gamma$, $\cos\gamma$ terms as was done in Ref. 26. This, however, will not be pursued in the present research. Next, with the assumption that x is monotonic, the independent variable is changed from time to range to obtain the intermediate vehicle model as

$$V' = \frac{g(T_0 + T_\eta - D)}{VW} - \frac{g}{V}\gamma \tag{1}$$

$$h' = \gamma \tag{2}$$

$$0 = \frac{g}{V^2}\left(\frac{L}{W} - 1\right) \tag{3}$$

$$\tilde{W}' = \frac{Q_0 + Q\eta}{V} \tag{4}$$

Here, V is airspeed, g acceleration caused by gravity, T_0 the idle thrust, $T_0 + T$ the maximum thrust, D aerodynamic drag, W vehicle weight, h altitude, x range, L lift, Q_0 the idle thrust fuel flow rate, $Q_0 + Q$ the maximum fuel flow rate, and $\tilde{W}$ the weight of fuel consumed. A prime over the variables indicates differentiation with respect to range x. Equation (3) implies that lift $\simeq$ weight. Hence, as in the energy-range model,[1,13,14] the drag in the intermediate vehicle model is evaluated with lift = weight. The control variables in this model are the throttle η and the flight-path angle γ. It is clear that the small path angle assumption is consistent with monotonicity of range assumption.

With range as the independent variable, time may be computed by evaluating the integral

$$t = \int_0^{x_f} \frac{dx}{V} \tag{5}$$

In order to compute the aerodynamic drag D, the lift coefficient C_L is first calculated using the condition lift = weight. The drag coefficient is next interpolated as a function of lift coefficient and Mach number. Models incorporating a quadratic dependence of C_L on C_D as well as a more general dependence have been considered in the present research.

The aircraft maximum thrust and fuel flow rate are assumed to be given as tabular functions of Mach number and altitude. The throttle parameter η is chosen to satisfy the constraint

$$0 \le \eta \le 1 \tag{6}$$

Thus, $\eta = 1$ would correspond to maximum thrust and fuel flow rate while $\eta = 0$ would yield the engine idle conditions. This type of throttle-thrust modeling is often because of the limited availability of experimental data. For the transport aircraft considered in this paper, it has been verified[27] that this linear thrust-throttle relationship gives rise to about 10% error in thrust at low throttle setting, and about 3% error in the fuel flow rate near the 50% throttle setting.

Note that this model incorporates the linear thrust-throttle assumption. In all that follows, it will be assumed that the aircraft weight is constant, because, unlike rockets, the effect of weight variations on aircraft dynamics is insignificant.[22,23] In addition to the dynamic equations, the system is subjected to other state-control constraints such as the terrain limit, Mach limit, and the maximum lift coefficient constraint. These will not be included in the ensuing.

With this background on the vehicle modeling, the next section will define the optimal aircraft cruise problem and the first-order necessary conditions will be derived.

The Optimal Control Problem

With intermediate vehicle modeling, the optimal aircraft cruise problem in the Lagrange form[28] is as follows: Determine the peicewise continuous control functions η,γ that yield

$$\min_{\eta,\gamma} \int_0^{x_f} \frac{Q\eta}{V}\,\mathrm{d}x \tag{7}$$

subject to the differential constraints (1) and (2) and the specified inequality constraints. The initial conditions and terminal conditions are free. In the performance index (7), x_f is a large final range. Note that the idle throttle fuel consumption has been dropped from the performance index. In all that follows, the idle thrust and the idle fuel flow rate will be dropped from consideration primarily to simplify the analysis. The present analysis, however, can take these into account without difficulty.[27] Indeed, for the transport aircraft used in the present study, these terms will be explicitly included in the analysis. The trajectories within the level flight envelope will be considered here since the assumption of lift = weight is an essential part of the intermediate vehicle modeling. Note that the solutions emerging from this analysis should satisfy the assumptions of small γ and γ'. Otherwise the intermediate vehicle modeling assumptions will be violated and the results will have doubtful validity.

With the foregoing definition of the cruise problem, the necessary conditions for optimality may be derived using Pontryagin's minimum principle.[8,27] Alternately, the optimal control problem can be transformed into the classical calculus of variations form[29] and Euler's necessary condition derived. The chief motivation to go through this transformation is that the aircraft cruise problem has a nice characterization in this form. Both approaches are equivalent and yield the same final form for the Euler's necessary conditions whenever the controls are within their bounds.

Subtituting for γ in Eq. (1) from expression (2), with T_0 and Q_0 equal to zero, and after some manipulations, one has

$$\eta = \frac{D}{T} + \left[V' + \frac{g}{V}h'\right]\frac{WV}{gT} \tag{8}$$

Substituting for η in Eq. (7) from Eq. (8) results in the following:

$$\min_{V,h} \int_0^{x_f} \left\{\frac{QD}{VT} + \frac{QW}{gT}V' + \frac{QW}{VT}h'\right\}\mathrm{d}x \tag{9}$$

Thus the calculus of variations problem can be stated as: Determine piecewise smooth trajectories $V(x)$ and $h(x)$ such that the integral (9) is minimized.

Equation (9) has an interesting interpretation. The first term in the integrand depends only on airspeed and altitude, while the second and third terms depend additionally on airspeed and altitude rates, respectively. Therefore, the first term in the integrand may be thought of as the cost associated with static paths; i.e., paths with $V' = h' = 0$. Indeed, this term is minimized in the classical steady-state cruise analysis for determining the fuel optimal cruise arc. The second and third terms constitute the dynamic part of the total cruise cost. If trajectories $V(x)$ and $h(x)$ can be found such that the sum of the second and third terms have a net negative sign over the integration interval, then that path will provide a lower cost when compared with a static path. In all that follows, it will be assumed that the integrand (9) has continuous partial derivatives with respect to V and h.

The calculus of variations problem defined in expression (9) is identically nonregular[29] since the Legendre-Clebsch necessary condition will be met only in the weak form. Other second-order necessary conditions such as the Jacobi test[29] are not directly applicable in these problems. Moreover, the extremals emerging may not satisfy all the boundary conditions. This is because of the degenerate[30] Euler's necessary conditions in identically nonregular problems. For instance, the minimum time airplane climb problem with the energy state model is degenerate by two orders.[31] Note that the throttle constraint in the transformed problem appears as

$$\eta_{\min} \le \frac{D}{T} + \left[V' + \frac{g}{V}h'\right]\frac{WV}{gT} \le \eta_{\max} \tag{10}$$

It may be noted that the control variable inequality constraint in the original optimal control problem is transformed into an inequality constraint on the states and their rates in the calculus of variations problem. In order to compute bang-bang control arcs, the boundary of the admissible region can be determined by using the inequality (10). Since this is an inequality constraint, within an admissible region, the optimal control problem may be analyzed without the explicit inclusion of this constraint.

The Euler's necessary conditions for the calculus of variations problem (9) can be derived as[29]:

$$h'\left\{\frac{1}{g}\frac{\partial}{\partial h}\left[\frac{Q}{T}\right] - \frac{\partial}{\partial V}\left[\frac{Q}{VT}\right]\right\} = \frac{1}{W}\frac{\partial}{\partial V}\left[\frac{QD}{VT}\right] \tag{11}$$

$$V'\left\{\frac{1}{g}\frac{\partial}{\partial h}\left[\frac{Q}{T}\right] - \frac{\partial}{\partial V}\left[\frac{Q}{VT}\right]\right\} = \frac{-1}{W}\frac{\partial}{\partial h}\left[\frac{QD}{VT}\right] \tag{12}$$

These necessary conditions hold between corners in the Euler solution. If one expect large number of corners in the solution, it is advisable to use the integral form of Euler's necessary conditions. However, in that case, more sophisticated numerical algorithms will be necessary to obtain the solutions. In all that follows, it will be assumed that the corners in the solution occur only when the Euler solution includes a piece of the boundary of the admissible region, i.e., when the singular arc is joined with a throttle saturation arc. The expressions (11) and (12) yield extremals for the aircraft cruise problem whenever the controls are within their bounds. If the quantity within the braces on the left-hand side of these expressions is not zero, it may be moved to the denominator on the right-hand side to obtain the Euler's necessary conditions in explicit form. It is interesting to note that this quantity is the expression for the Goh-Robbins test, first obtained by Speyer.[3] Speyer has shown that this expression is nonzero, in general. If this expression were zero, one would obtain two implicit algebraic expressions defining the classical steady-state cruise point for certain aircraft. This case, however, admits multiple solutions. This is because, as long as the differential constraints in the problem are satisfied, any h' and V' functions sufficiently fast such that the aircraft stays substantially in the vicinity of the classical steady-state cruise point will satisfy Euler's necessary conditions.

The Euler's necessary conditions, including idle thrust and idle throttle fuel flow rate, can be obtained simply by replacing the term QD/VT on the right-hand side of expressions (11) and (12) by $(Q_0/V)+[Q(D-T_0)/VT]$. The time-constrained aircraft cruise problem may be analyzed in an identical setting.[27]

As noted earlier, the Euler equations (11) and (12) are degenerate[30] by one order each, because in regular problems these necessary conditions are second-order differential equations.[29] The implication is that the resulting extremals can meet only one boundary condition on altitude and airspeed. This is not of undue concern in the present analysis since the cruise arc forms the interior arc and transients to and from this arc can be constructed subsequently. On the other hand, the degeneracy is an advantage since the necessary Euler conditions can now be evaluated using an initial value integration scheme. It may be noted in passing that no extremal emerging from the necessary Euler conditions (11) and (12) can be made to satisfy the natural boundary conditions. It can be shown that every point within the level flight envelope satisfies the Weierstrass-Erdmann corner conditions. Moreover, the Weierstrass excess function[29] is identically zero in the entire admissible region.

The necessary Euler conditions are two first-order, homogeneous, nonlinear ordinary differential equations for which explicit analytical solutions are difficult to construct. However, analysis in the small can be carried out using the phase plane—in this case, the altitude-airspeed chart. It is clear that the singular point of the system (11), (12) is at

$$\frac{\partial}{\partial V}\left[\frac{QD}{VT}\right]=0,\quad \frac{\partial}{\partial h}\left[\frac{QD}{VT}\right]=0 \tag{13}$$

Interestingly, the expressions in (13) are the necessary conditions for a proper interior minimum or maximum of the function QD/VT. In the classical steady-state analysis, these conditions define the cruise point. However, it will be seen later in this paper that the function QD/VT may not have a proper interior minimum for certain aircraft. In that case, the singular point of the necessary Euler conditions and the classical steady-state cruise point will not coincide. This factor has important bearing on the nature of optimal cruise arc emerging from the present analysis.

Additionally, it can be verified that the quantity QD/VT is a constant of motion along each solution produced by the necessary Euler conditions (11), (12). Though this fact does not make the task of obtaining Euler solutions any easier, it is a useful device for monitoring the accuracy of numerical solutions.

As is well known from nonlinear system theory, a family of solutions will exist in a region about the singular point, the character of the paths being determined by the nature of the singular point. In particular, if the singular point were a vortex or a center, the family of solutions form closed trajectories around the singular point. This forms the basis for deriving a test for the existence of oscillatory extremals given in an ensuing section.

Before embarking upon the development of such a test, the boundary of the admissible region defined by the transformed throttle constraint, viz, the expression (10), will be investigated. With $\eta_{min}=0$ substituting for V' and h' from the necessary Euler conditions (11) and (12), the zero throttle boundary of the admissible region can be obtained as:

$$\left.\frac{\partial D}{\partial h}\right|_{E=\text{const}}=0 \tag{14}$$

where $E=h+(V^2/2g)$ is the specific energy. This expression defines the maximum range glide trajectory in the altitude-airspeed plane with intermediate[26] or energy-modeled aircraft. This result is in accord with the work of Gilbert and Parsons[13] and Houlihan et al.[14]

Similarly, for $\eta=1$, it can be shown that

$$\left\{\frac{-1}{T}\right\}\left.\frac{\partial D}{\partial h}\right|_{E=\text{const}}=\left.\frac{\partial}{\partial h}\left\{\ell n\left[\frac{Q}{VT}\right]\right\}\right|_{E=\text{const}} \tag{15}$$

Expression (15) defines yet another trajectory in the altitude-airspeed plane. However, this path does not appear to have any direct interpretation in terms of the optimal trajectories reported in the literature.

If the boundaries of the admissible region in the altitude-airspeed plane intersect at least at two points, one can hope to obtain a pure bang-bang arc to be the outermost member of the Euler solution family. The requirement for at least two intersections is dictated by the long-range nature of the cruise problem which demands that if a power-off arc exists, it must be followed by a power-on arc in order to maintain energy in the system. Expressions (14) and (15) will have to be simultaneously satisfied at those points. This implies

$$\left\{\frac{1}{g}\frac{\partial}{\partial h}\left[\frac{Q}{T}\right]-\frac{\partial}{\partial V}\left[\frac{Q}{VT}\right]\right\}=0 \tag{16}$$

The quantity on the left-hand side can be identified as the Goh-Robbins test[3] for the cruise problem with intermediate vehicle modeling. Thus, it has been shown that with intermediate vehicle modeling, the Goh-Robbins test will have to be met at least at two points within the aircraft flight envelope for a pure bang-bang arc to be the limiting member of the Euler solution family.

Such points will not be found within the aircraft flight envelope for the three aircraft considered in this paper. In the next section, it will be shown through numerical computations that the admissible region is closed only by the level flight envelope on one side and by the minimum altitude constraint on the other. The implication is that the optimal control settings emerging from current modeling are at best of the saturating type, viz, partial throttle setting throughout or partial throttle followed by a control saturation. For the scalar control case, such saturating types of optimal control arcs are known to arise whenever the Kelley-Contensou test is met for odd q; see Kelley et al.[32] for details. For the vector control case, a similar test is not currently available.

It is important to stress here that the effect of a flight-path angle constraint has not been considered in the present research. Imposition of such a constraint may alter the foregoing conclusions.

The Existence of Oscillatory Extremals

A test for the existence of oscillatory extremals can be constructed by linearizing the necessary Euler conditions (11) and (12) about the singular point of the system and examining the nature of the roots of the characteristic polynomial. If the roots turn out to be purely imaginary, the solutions in the neighborhood will be oscillatory. Note that this test is applicable only if a singular point exists within the aircraft level flight envelope.

From this analysis, the condition for roots of the characteristic equation to be purely imaginary turns out to be

$$\frac{\partial^2\mu}{\partial^2 V}\frac{\partial^2\mu}{\partial^2 h}-\left[\frac{\partial^2\mu}{\partial V\partial h}\right]^2>0, \qquad \mu=\frac{QD}{VT} \tag{17}$$

Interestingly, expression (17) is a sufficient condition for the singular point defined by expressions (13) to provide a proper interior minimum of the function QD/VT; provided that

$$\frac{\partial^2\mu}{\partial^2 V}>0, \qquad \text{or} \qquad \frac{\partial^2\mu}{\partial^2 h}>0 \tag{18}$$

Under the conditions of (17) and (18), the Hessian of QD/VT with respect to V and h will be positive definite. This leads to the conclusion that, if the singular point were a proper interior minimum of the function QD/VT, then oscillatory extremals will always exist in the neighborgood of this point. In this case, the singular point also would be the classical steady-state cruise point.

This is likely to happen only if the classical steady-state cruise point were well within the level flight envelope of the aircraft. In the immediate vicinity of the singular point, the period of the oscillatory Euler solution is given by

$$x_c=\frac{2\pi}{\omega_c} \tag{19}$$

where

$$\omega_c=\frac{1}{W\Delta}\sqrt{\frac{\partial^2\mu}{\partial^2 V}\frac{2^2\mu}{\partial^2 h}-\left[\frac{\partial^2\mu}{\partial V\,\partial h}\right]^2} \tag{20}$$

Here x_c is the range for one oscillation and

$$\Delta=\left\{\frac{1}{g}\frac{\partial}{\partial h}\left[\frac{Q}{T}\right]-\frac{\partial}{\partial V}\left[\frac{Q}{VT}\right]\right\}$$

The existence test obtained in the foregoing assumed that a singular point exists within the aircraft level flight envelope. For certain aircraft there may not be a singular point within the level flight envelope. In this case, an alternate test form of the existence test can be devised[27] using Bendixson's theorem, albeit in somewhat less elegant form. This, however, will not be pursued in the present paper.

Numerical Results

The analysis presented so far will be employed next to explore the nature of Euler solutions for three aircraft configurations, viz, a high-performance fighter aircraft,[16] a version of the F-4 aircraft[14,26] without afterburner, and a short-haul transport aircraft.[22,23] In all that follows, functions of one variable, such as the aerodynamic coefficients, are interpolated using cubic splines. Cubic spline lattices[26] are used to interpolate functions of two variables such as engine thrust and fuel flow rate given as functions of Mach number and altitude. Euler's necessary conditions are integrated using a fourth-order Kutta-Merson technique with variable step size. The accuracy of numerical solutions are monitored using the constant of motion QD/VT. Along every Euler solution given here, this quantity is maintained constant within 10^{-14}. All the computations are carried out with double-precision arithmetic on a VAX 11/750 machine.

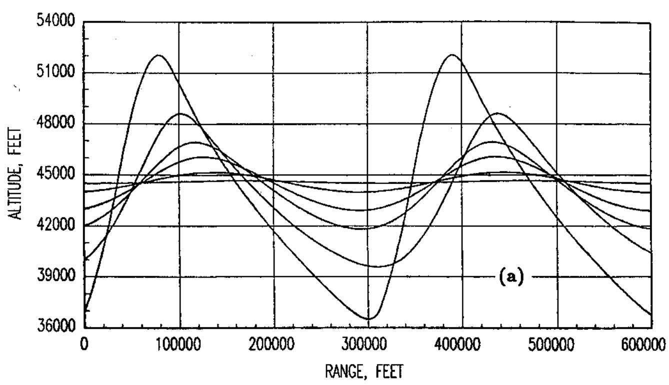
High-Performance Fighter Aircraft

The classical steady-state cruise point for this aircraft is determined by first obtaining the airspeeds satisfying

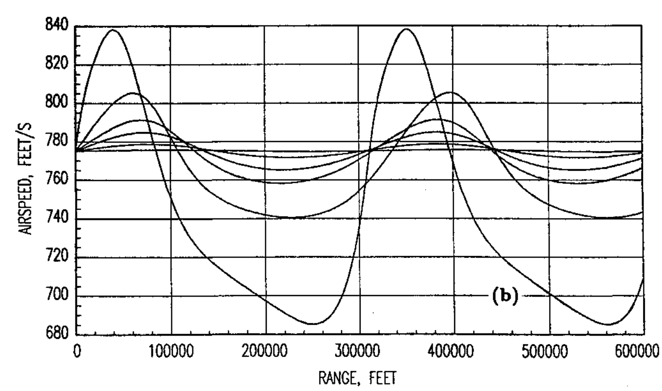
$$\frac{\partial}{\partial V}\left[\frac{QD}{VT}\right]=0 \tag{21}$$

at several altitudes within the level flight envelope. The altitude-airspeed pair providing the least fuel consumed per unit range is then the classical steady-state cruise point. For this aircraft, cruise point occurs within the level flight envelope, at approximately 44,580 ft altitude and 775.25 ft/s airspeed. The test for the existence of oscillatory extremals is satisfied at the classical steady-state cruise point. The period of oscillatons in the vicinity of the classical steady-state cruise point predicted using expression (20) turns out to be about 57.8 mi.

Next, the unconstrained necessary Euler conditions (11) and (12) are integrated, starting at several altitude-airspeed initial conditions. The results of this study are presented in Fig. 1. If the initial conditions chosen are close to the steady-state cruise point, the altitude-airspeed trajectories are nearly sinusoidal. As one moves away from the steady-state cruise point, the oscillations are characterized by a relatively steep ascent followed by a long descent trajectory. The maximum amplitude of altitude oscillation permissible within the level flight envelope appears to be about 17,000 ft, the corresponding airspeed amplitude is about 175 ft/s. Beyond this, the oscillatory trajectories can be formed only by the inclusion of a part of the level flight envelope. Along the largest amplitude oscillation, the maximum flight-path angle is about 20 deg and the minimum about −7 deg. The throttle along this trajectory varies between −1.02 and 3.43. Note that these solutions did not include the throttle constraint.



a) Range vs altitude



b) Range vs airspeed

Fig. 1 Unconstrained Euler solutions for the high-performance fighter.

In Fig. 2, some Euler solutions are superimposed on the level flight envelope along with the boundary of the admissible region and the locus of steady-state cruise points. In this figure, the admissible region in the altitude-airspeed plane is the area bounded by the curve LMNOL. This figure reveals that the zero-throttle boundary does not intersect the maximum throttle boundary within the aircraft flight envelope. Thus, with intermediate vehicle modeling, for this aricraft, the throttle will not operate in a pure bang-bang mode along the limiting member of the Euler solution family.

Additionally, from Fig. 2, it can be seen that the outermost solution violates both the zero-throttle constraint and the maximum throttle constraint. The innermost solution is well within the admissible region while an intermediate solution appears to violate only the maximum throttle constraint. At this juncture, an examination of the necessary Euler conditions reveals that the nature of throttle constraint violation along a particular solution is entirely governed by the slope of the quantities Q, D, and T in the altitude-airspeed plane. Within the admissible region, the flight path angle along every Euler solution is within ± 3 deg.

Since the family of Euler solutions forms closed orbits in the altitude-airspeed plane, each member is a candidate extremal for long-range cruise. Selection of a particular member of this family can only be based on the amount of fuel it consumes to cover a unit range when compared with classical steady-state cruise. With this objective, the fuel consumed per oscillation is computed and divided by the corresponding range covered to obtain the fuel consumed per unit range. The fuel consumed to cover a unit range along the classical steady-state cruise arc is next subtracted from this quantity and the percent of fuel lost or gained in employing a particular member of the extremal family is computed. Since these numbers can be very small, sometimes close to the machine precision, these calculations are carried out for several integration step sizes.

Within the permissible throttle limits, the maximum fuel savings appear to be of the order of $1.84 \times 10^{-4}\%$, corresponding to about 766.74 ft altitude oscillation and throttle oscillating from 0.55–0.69. It is to be emphasized here that the nature of the throttle constraint violation depends entirely on the slope of the function QD/VT in the altitude-airspeed space. As a result, for some other aircraft, a much larger oscillation amplitude might be permissible without violating the throttle constraints.

Before closing this section, it is worthwhile recalling that earlier research[14] showed that the ratio of maximum thrust to minimum drag controls the amount of fuel savings realized in employing chattering cruise. The consequences of increasing the aircraft thrust in the present case would be an expanded flight envelope along with the movement of the maximum throttle boundary toward the right in the altitude-airspeed chart. The combined effect would be an increase in the admissible region, permitting larger oscillation amplitude. If the aircraft L/D is also simultaneously increased, the maximum range glide locus would move to the left in the altitude-airspeed plane, further increasing the admissible region. With this, the high performance fighter may show larger fuel savings.

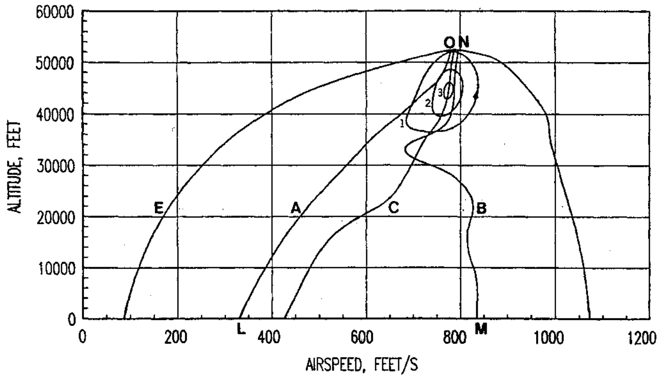


Fig. 2 Euler solutions for the high-performance fighter: A) minimum throttle boundary; B) maximum throttle boundary; C) locus of steady-state cruise points; E) level flight envelope; and 1,2,3) Euler solutions, LMNOL: boundary of the admissible region.

F-4 Aircraft

For this aircraft, classical steady-state cruise point occurs well within its level flight envelope, at approximately 32,650 ft altitude and 812 ft/s airspeed. The condition for the existence of oscillatory extremals is satisfied at this point. The predicted period turns out to be about 29.6 miles. The Euler solutions for this aircraft are characterized by a relatively steep ascent followed by a shallow descent. The period of the largest permissible oscillatory trajectory for this aircraft is about 32.007 miles; little more than half the period for the high performance fighter. The fuel consumed along the longest permissible oscillatory trajectory is about 0.9% lower than that long the classical steady-state cruise.

At this point, it is perhaps worthwhile to compare the present results with those of Grimm et al.[20] In that work, they appear to have used similar aircraft data to compute oscillatory cruise trajectories. Comparisons between their results and the present work are summarized in Table 1. From this table, it can be seen that the results are close despite the differences in the throttle history, viz, in the present work the throttle oscillates between the limits, while in Ref. 20 the throttle always operates in a bang-bang mode. The present analysis tends to underpredict the fuel savings, however.

Transport Aircraft

Unlike the fighter aircraft, the transport aircraft engine data is available at several engine pressure ratio (EPR) settings including maximum and idle.

As before, classical steady-state cruise point for this aircraft is calculated using expression (21) at several altitudes. This calculation showed that the classical steady-state cruise point does not provide a proper interior minimum for the function QD/VT. A minimum exists, however, but it occurs on the boundary. The cruise condition for this aircraft turns out to be at about 37,500 ft altitude and 740 ft/s airspeed. Thus, the classical steady-state cruise point for this aircraft is not the singular point of the Euler necessary conditions. Moreover, since the classical steady-state cruise point lies on the flight envelope, the cruising flight occurs at full throttle. The following conditions hold at the classical steady-state cruise point:

$$\frac{\partial}{\partial V}\left[\frac{QD}{VT}\right] = 0, \qquad \frac{\partial}{\partial h}\left[\frac{QD}{VT}\right] < 0 \tag{22}$$

Table 1 Comparison of a Euler solution for the F-4 aircraft

	Present work	Grimm et al.[20]
Amplitude of altitude oscillation	5913 ft	7557.5577 ft
Amplitude of airspeed oscillation	40.98 ft/s	124.47736 ft/s
Oscillation period	32.007 mi	32.8 mi
Fuel per mile saved	0.9%	≈2%
Nature of throttle history	Continuously varying between the given bounds	Bang-bang

Difference in classical steady-state cruise conditions (Ref. 20–present)
$\Delta V =$ 10.788 ft
$\Delta h = 4084.662$ ft

No singular point has been found within the level flight envelope of this aircraft. As a result, the existence test (17) is no longer applicable.

As before, Euler solutions are generated starting from several initial conditions within the level flight envelope. In Fig. 3, these solutions are superimposed on the level flight envelope along with the boundary of the admissible region and the locus of the steady-state cruise points. From this figure it can be seen that there are no oscillatory solutions lying entirely within the flight envelope. The singular point of the Euler solution family appears to be somewhere outside the level flight envelope. Within the envelope, only segments of the oscillatory solution exist. Indeed, given sufficient range, all the trajectories tend to go outside the level flight envelope.

The idle-throttle constraint boundary, the maximum throttle constraint boundary, and the locus of steady-state cruise points are also given in Fig. 3. The region enclosed by the contour LMNL and the line segment NO constitute the admissible region. An interesting part of this admissible region is the line segment NO. All the points on the line segment NO satisfy the Goh-Robbins test since the maximum and minimum throttle boundaries are colinear. If the necessary Euler conditions were to be integrated after including throttle constraints and a constraint on the flight-path angle, starting from an altitude-airspeed initial condition on this line, a chattering arc would result.

In view of the foregoing discussion, none of the unsteady Euler solutions are candidates for long-range cruise. These paths form range-fuel optimal trajectories connecting various points within the flight envelope. Thus, for the transport aircraft, the optimal long-range cruise trajectory is a full throttle flight at the classical steady-state cruise point. The classical steady-state cruise point turns out to be the maximum altitude point on the level flight envelope. This conclusion agrees with that of Hargraves et al.[24] and also with that of Barman and Erzberger.[22]

The solutions presented in this paper are all unconstrained, i.e., the throttle constraint was not explicitly included in the construction of Euler solutions. The effect of throttle constraint and an altitude constraint on the high-performance fighter and the transport aircraft have been evaluated; see Ref. 27 for details.

Second Variation Analysis

It is known[29] that the sufficient conditions for a functional $J(x)$ to have a weak local minimum for $x = x^*$ are that the first variation $\delta J(x)$ vanishes for $x = x^*$, and that the second variation $\delta^2 J(x)$ be strongly positive for $x = x^*$. The conditions for vanishing of the first variation leads to Euler-Lagrange equations and transversality conditions. The positivity requirement of the second variation leads to the Legendre-Clebsch and Jacobi's necessary condition for regular extremals. In addition to these, if Weierstrass's necessary condition is satisfied in the strong form, the extremal affords a strong local minimum for the functional $J(x)$.

For singular extremals, the Legendre-Clebsch necessary condition is satisfied only in the weak form. In this case, the Legendre-Clebsch necessary condition is replaced by the generalized Legendre-Clebsch necessary condition. The Jacobi test in its original form is not applicable for these problems. An alternate approach for obtaining second-order necessary conditions involves the use of transformations proposed by Kelley[33] and Goh.[11]

In this section, an expression for the second variation for the aircraft cruise problem in the vicinity of the classical steady-state cruise point will be developed. This expression imbeds both the generalized Legendre-Clebsch necessary condition[3] and the condition for the existence of oscillatory extremals.

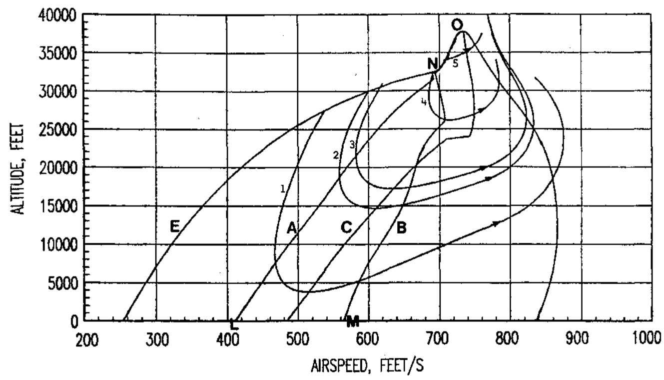


Fig. 3 Euler solutions for the transport aircraft: A) minimum throttle boundary; B) maximum throttle boundary; C) locus of steady-state cruise points; E) level flight envelope; and 1–5) Euler solutions, LMNL-NO: boundary of the admissible region.

In the section on transport aircraft, it was shown that the steady-state cruise is a full throttle arc occurring at the maximum altitude on the flight envelope. Consequently, a second-order test cannot be set up in the usual sense for this aircraft because only unilateral variations in altitude are permissible at the classical steady-state cruise point. However, it is clear that an unconstrained Euler solution originating from any point within the level flight envelope, arbitrarily close to this classical steady-state cruise point, will rapidly approach the flight envelope and tend to stay on it. This is because the energy rate is zero along the flight envelope. Hence, the remainder of this section will be devoted to the second variation analysis of the fighter aircraft.

Second Variation Near the Classical Steady-State Cruise Point

Though the aircraft cruise problem is not strictly a fixed end point problem, if the integration interval is chosen as the length of a full cycle of oscillation as in Ref. 17, it may be treated as a fixed end point problem. Additionally, in strict terms, it is not correct to compare various members of the Euler solution family because each of them have a unique set of boundary conditions and, possibly, different periods of oscillation. In the vicinity of the classical steady-state cruise point, however, the cost of moving from one set of boundary conditions to the next is much smaller than the long-range cruise cost. Further, in this region, the differences in the frequency between extremals is small.

With these approximations, the second variation[29] for the aircraft cruise problem in the vicinity of the classical steady-state cruise is given by

$$\delta^2 J = \int_0^{x_p} 2\omega(\delta V, \delta h, \delta V', \delta h')\, dx \tag{23}$$

where

$$\begin{aligned} 2\omega = {} & \mu_{VV}\,\delta V^2 + \mu_{hh}\,\delta h^2 + 2\mu_{Vh}\,\delta V\,\delta h + 2f_{VV'}\,\delta V\,\delta V' \\ & + 2f_{hh'}\,\delta h\,\delta h' + 2f_{Vh'}\,\delta V\,\delta h' + 2f_{V'h}\,\delta V'\,\delta h \end{aligned} \tag{24}$$

Since the second partial derivatives used in expression (24) are evaluated at the classical steady-state cruise point, they can be treated as constants. Moreover, the terms containing $\delta V'\delta V$ and $\delta h'\delta h$ vanish at the integration limits because these are the derivatives of the expression $\frac{1}{2}(\delta V)^2$ and $\frac{1}{2}(\delta h)^2$, and

$$\delta V(0) = \delta V(x_p), \qquad \delta h(0) = \delta h(x_p) \tag{25}$$

The last term in expression (24) can be integrated next by parts to yield the following expression for second variation:

$$\delta^2 J = \int_0^{x_p} [\delta V\, \delta h] \begin{bmatrix} \mu_{VV} & \mu_{Vh} \\ \mu_{hV} & \mu_{hh} \end{bmatrix} \begin{bmatrix} \delta V \\ \delta h \end{bmatrix} \mathrm{d}x - \int_0^{x_p} [2f_{V'h} - 2f_{Vh'}]\, \delta V\, \delta h'\, \mathrm{d}x \tag{26}$$

It is clear that the first term of the second variation (26) will be positive for arbitrary piecewise smooth variations $\delta V, \delta h$, not identically zero and vanishing at the two ends, if the 2×2 matrix in the integrand were positive definite. This matrix will be positive definite if the classical steady-state cruise point provides an interior minimum for the function QD/VT. It was shown earlier that oscillatory Euler solutions will exist under precisely the same conditions.

Substituting the required second partial derivatives in the second term in Eq. (26), one has

$$[2^{\text{nd}} \text{ term}] = -2W \int_0^{x_p} \left\{ \frac{1}{g} \frac{\partial}{\partial h} \left[\frac{Q}{T} \right] - \frac{\partial}{\partial V} \left[\frac{Q}{VT} \right] \right\} \delta V\, \delta h'\, \mathrm{d}x \tag{27}$$

The term within the braces can be recognized as the denominator of the Euler's necessary conditions. More significantly, it is the Goh-Robbins test for the aircraft cruise problem with the intermediate vehicle model, first obtained by Speyer[3] along the classical steady-state cruise arc and rederived for the present Euler solution family by Gracey.[27] According to Speyer and Gracey, a necessary condition for the second variation to be nonnegative is that the term within the braces in Eq. (27) should be zero.

With numerical calculation, it is possible to show that for the aircraft used in the present study, this term is greater than zero. Furthermore, if the extremals emerging are such that the product $\delta V\, \delta h'$ is greater than zero, this term will be negative.

Before proceeding to test the sign of the second variation in the vicinity of the classical steady-state cruise arc, the connection between the present research and that of Breakwell and Shoee[17] will be examined. Carrying out the partial differentiation indicated in Eq. (27), and putting $\sigma_c = Q/T$, $V_c = V$, i.e., the thrust specific fuel consumption and the airspeed at the classical steady-state cruise point, one has,

$$[2^{\text{nd}} \text{ term}] = -2W \int_0^{x_p} \left\{ \frac{1}{g} \frac{\partial}{\partial h} \left[\frac{Q}{T} \right] - \frac{1}{V} \frac{\partial}{\partial V} \left[\frac{Q}{T} \right] \right\} \delta V\, \delta h'\, \mathrm{d}x - 2 \int_0^{x_p} m \frac{\sigma_c g}{V_c^2}\, \delta V\, \delta \gamma\, \mathrm{d}x \tag{28}$$

Expression (28) has used the fact that $\delta h' = \delta \gamma$, the linearized h' in Eq. (2).

The second term in expression (28) is identical to the condition obtained by Breakwell and Shoee,[17] using a point-mass model for aircraft flight. According to them, this term is responsible for fuel savings along oscillatory trajectories, provided that δV and $\delta \gamma$ have the same sign along such paths. Along every oscillatory extremal generated for the fighter aircraft, the quantity $\delta V\, \delta \gamma$, generated by subtracting the classical steady-state cruise conditions from a Euler solution, turns out to be positive. Thus, the present analysis reveals that Breakwell and Shoee's condition[17] is a component of the Goh-Robbins test with the intermediate vehicle modeling.

Because of the singular nature of the aircraft cruise problem, it is difficult to establish the sign of second variation for arbitrary PWS variations. Nevertheless, according to Kelley and Moyer,[34] the variations obtained from the linearized necessary Euler conditions can be used to verify the sign of second variation, since the competition, if any, is between neighboring extremals of the original problem. Thus, if a system of nonzero variations can be found that makes the second variation zero, then it is clear that a neighboring path is competitive and that the test extremal furnishes at best an improper minimum of J and at worst a merely stationary value.

The variations δV, δh, $\delta V'$, $\delta h'$ are generated in terms of the initial conditions $\delta V(0)$, $\delta h(0)$ by solving linearized Euler equations about the classical steady-state cruise point. These variations are used next to carry out the indicated integrations in expression (26). Applying the given integration limits to these expressions and normalizing by the range covered per oscillation, it is possible to show

$$\delta^2 J = -\frac{1}{2} \left[1 - \frac{\Delta^2}{\Delta_a^2} \right] [\delta V(0)\ \delta h(0)] \begin{bmatrix} \mu_{VV} & \mu_{Vh} \\ \mu_{hV} & \mu_{hh} \end{bmatrix} \begin{bmatrix} \delta V(0) \\ \delta h(0) \end{bmatrix} \tag{29}$$

Here, Δ_a is the quantity Δ evaluated along an arbitrary neighboring extremal. The second variation on an extremal in the vicinity of the classical steady-state cruise point is given by expression (29). Note that it will be negative for arbitrary initial conditions $\delta V(0)$ and $\delta h(0)$ not simultaneously zero, if the 2×2 matrix on the right-hand side is positive definite; and if the inequality

$$\Delta^2 / \Delta_a^2 < 1 \tag{30}$$

is satisfied. The 2×2 matrix on the right-hand side of expression (29) is the Hessian of the function QD/VT with respect to V and h. Earlier in this paper it was shown that this matrix will be positive definite whenever oscillatory solutions exist.

Since $\Delta = \Delta_a$ at the classical steady-state cruise point, the second variation is zero along this arc, thereby indicating that a neighboring extremal is competitive.

Moreover, if the function QD/VT is such that condition (30) is satisfied, then an oscillatory extremal will yield lower value of fuel consumption than classical steady-state cruise. It is important to note that the foregoing second variation analysis is valid only in the neighborhood of the classical steady-state cruise point. A more general second variations analysis can only be carried out numerically.

Conclusions

The optimal long-range aircraft cruise problem was analyzed in this paper using an intermediate vehicle model. Euler's necessary conditions were derived, and the nature of solutions was discussed. It was shown that the classical steady-state cruise point is the central member of the Euler solution family for certain aircraft. The nature of minimum throttle arc and the maximum throttle arc was investigated. The minimum throttle arc turns out to be the maximum range glide path. An expression for the maximum throttle arc was obtained. A test for the existence of oscillatory cruise trajectories was developed.

The present analysis indicates that, depending on the aircraft configuration, one could have the classical steady-state cruise point within the flight envelope or on it. In the former case, the question of operation at the classical steady-state cruise point versus that along oscillatory trajectories occurs naturally. In the latter case that question is artificial since a competing extremal can be set up only by imposing additional constraints. Numerical studies with three aircraft configurations indicate that, for the fighter aircraft, a family of oscillatory solutions exists about the classical steady-state cruise point. Some members of this Euler solution family yield lower values of fuel consumption when compared with the classical steady-state cruise. It appears that a large thrust combined with a high lift-over-drag ratio is conducive to the existence of oscillatory cruise solutions and also to increased fuel savings along such extremals. For the transport aircraft, the classical steady-state cruise point occurs on the level flight envelope. No competing oscillatory cruise extremals have been found for this aircraft.

The nature of the second variation in the vicinity of the classical steady-state cruise point was discussed. It was shown that the second variation can be decomposed into two components, the first of which will be positive whenever oscillatory solutions exist, while the second component will be negative whenever the Goh-Robbins test is not satisfied at the classical steady-state cruise point, and a condition similar to that of Breakwell and Shoee is satisfied. The term responsible for fuel savings along oscillatory extremals identified by Breakwell and Shoee, using a point-mass model, was shown to emerge in the present expression for the second variation.

Using the variations generated by the linearized Euler equations, it was shown that the second variation is zero at the classical steady-state cruise point.

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